ACCP vs. Atomicity in Additive Monoids of Cyclic Semirings

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Abstract & Timeline

- **1921** — Noether introduces **Ascending Chain Condition** (ACC)
- **1968** — Cohn modifies definition to use principal ideals (ACCP) and incorrectly asserts **ACCP** $\Leftrightarrow$ **atomic** for integral domains
- **1974** — Grams provides counterexample, implies **ACCP** $\subsetneq$ **atomic**
- **1982, 1993, 2019, 2022, 2023, 2025** — Further counterexamples
- **1992** — Halter-Koch extends terms to **commutative monoids**
- **2022** — Correa-Morris and Gotti explore **cyclic semirings**

We characterize atomicity and the ACCP in the class of additive monoids of cyclic semirings via algebraic conditions on α, involving its conjugates, algebraic integrality, and Perron-Frobenius theory.

Additive Monoids of Cyclic Semirings

$\mathbb{N}_0 =$ nonnegative integers. $\mathbb{N}_0[x] =$ polynomials over $\mathbb{N}_0$. For $\alpha \in \mathbb{C}$, $\mathbb{N}_0[\alpha] =$ polynomials in α with coefficients from $\mathbb{N}_0$,

$$\mathbb{N}_0[\alpha] := \{f(\alpha) \mid f \in \mathbb{N}_0[x]\} \subseteq \mathbb{C}.$$

$M_\alpha =$ underlying additive monoid of this cyclic semiring.

Algebraic Numbers & Minimal Polynomials

A complex number is **algebraic** if it is a root to a nonzero polynomial with rational coefficients, and **transcendental** otherwise.

Fact. If ρ is transcendental, then M_ρ has unique factorization.

If α is algebraic ($\alpha \in \mathbb{A}$), it satisfies a **minimal polynomial** $m_\alpha \in \mathbb{Q}[x]$. Other roots to this polynomial are called **algebraic conjugates** of α.

Fact. If α and β are algebraic conjugates, then $M_\alpha \cong M_\beta$.

Fact. If α has no positive conjugates, then M_α forms a group.

Atomicity & Antimatterness

- **Atom** — noninvertible element with no proper nonzero factors
- **Atomic** — (each non-unit) can be written as a finite sum of atoms
- **Antimatter** — no atoms

Fact. For $\alpha \in \mathbb{A}_{>0}$, M_α is precisely one of antimatter or atomic.

Fact. If $\alpha > 1$, M_α is atomic (& ACCP). If $\alpha < 1$, atomic $\Leftrightarrow$ 1 is an atom.

If 1 is not an atom, we may write $1 = a_1\alpha + \cdots + a_n\alpha^n$. That is, $f(\alpha) = 0$ for some $f \in x\mathbb{N}_0[x] - 1$, called an **antimatter decomposition**.

- **ACCP** — no strictly ascending chains of principal ideals, or atomic and cannot subtract atoms from element arbitrarily many times

Atomicity Characterization

Theorem. For $\alpha \in \mathbb{A} \cap (0, 1)$, M_α is antimatter if and only if

- α has no positive conjugate aside from itself,
- α^{-1} is an algebraic integer,
- and α does not exceed any of its algebraic conjugates in norm.

Proof of Necessity

Descartes' Rule of Signs

#sign changes $\geq$ #positive roots. Since f has one sign change, then m_α has a single positive root, so α has no other positive conjugate.

Algebraic Integers & Reciprocal Polynomials

Roots to monic integer polynomials are called **algebraic integers**. Since α^{-1} is a root to f^* (reciprocal polynomial of f), whose negative is monic with integer coefficients, then α^{-1} is an algebraic integer.

Frobenius Companion Matrices & Perron-Frobenius Theorem

We show the companion matrix of f^* is the (weighted) adjacency matrix of some strongly connected digraph, making it irreducible. Then, the Perron-Frobenius Theorem guarantees α^{-1} is a Frobenius root, i.e., at least as large as the norm of each of its algebraic conjugates.

Proof (Sketch) of Sufficiency

Goal. $g, h \in \mathbb{Z}[x]$ so $ghm_\alpha \in x\mathbb{N}_0[x] - 1$ (antimatter decomposition).

1. Set $g = (x+1)^N$ for some large N so gm_α has a single sign change.
2. Express h using a homogeneous linear recurrence relation.
3. Use Cramer's rule + alternants to find an explicit form for h.
4. Show $(\alpha^{-1})^n$ ultimately dominates the sequence and conclude that h is eventually positive. By the form of h, the same holds for ghm_α.

Examples

Let $q = n/d \in \mathbb{Q} \cap (0, 1)$ in lowest terms. If $n = 1$ (unit fraction), M_q is not atomic as $1 = q + q + \cdots + q$ (d times). Indeed, from our theorem, q has no other algebraic conjugate and $q^{-1} = d$ is an integer (thus algebraic integer). In contrast, if $n > 1$, M_q is atomic and each power of q is an atom. However, M_q does not satisfy the ACCP because dq^n generates an ascending chain that does not stabilize. Alternatively, there is an infinite chain of strict divisions, $\cdots \mid dq^3 \mid dq^2 \mid dq \mid d$.

Consider $\alpha = \sqrt[d]{q}$, so $m_\alpha = x^d - q$. While α now has distinct algebraic conjugates, none are positive and each equal α in norm. Hence, M_α is atomic precisely when M_q is. This agrees with $M_\alpha \cong M_q^k$.

ACCP Characterization

Theorem. For $\alpha \in \mathbb{A} \cap (0, 1)$, M_α satisfies the ACCP if and only if α has a positive conjugate greater than 1.

Proof

Sufficiency

Let $\beta > 1$ be a positive conjugate of α. Then, $M_\alpha \cong M_\beta$. Moreover, M_β can be *listed increasingly*—each atom is at least 1, so M_β is *discrete*.

Necessity (Sketch)

We show $(x+1)^N m_\alpha$ has its negative coefficients in clusters around each βN (for positive conjugates β of α) for large N, similar to Pólya's theorem on positive forms. So, for some $f, g \in \mathbb{N}_0[x]$, $k \in \mathbb{N}_0$, we may express that product as $(x^k - 1)f + g$. Thus, $(\alpha^{kn} f(\alpha))_{n \geq 0}$ generates an ascending chain of principal ideals that does not stabilize.

Conclusion & Discussion

Let $\alpha \in \mathbb{A}_{>0}$ be the largest positive root to m_α.

Antimatter	Atomic (desired)
$\alpha < 1$, $\alpha \leq \lvert\beta\rvert$ for each algebraic conjugate β, and α^{-1} is an algebraic integer	**ACCP** $\alpha > 1$

Remark. Our characterization in terms of the dominant root connects to Perron-Frobenius theory (nonnegative matrices and spectral radii).

These conditions produce many monoids that satisfy atomicity but not the ACCP. [3] shows the ascent of atomicity to integral domains (via **monoid algebras**) for many M_α, so our work may expand the list of counterexamples to Cohn's original assertion. Future directions are exploring notions **between** atomicity and the ACCP and **factorization invariants** (quantifying deviation instead of providing labels alone).

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